PENRITH SELECTIVE HIGH SCHOOL

MATHEMATICS DEPARTMENT

Trial HSC Examination

2022

Year 12 Extension 1 Mathematics

General Instructions

- Reading time – 10 minutes.
- Working time – 2 hours.
- Write using black pen.
- Answer in the spaces provided.
- No correction tape/white out allowed.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.
- Write your NESA Number above and on each additional working booklet.
- Reference sheets is supplied with the examination.

Total Marks: 70

Section I – 10 marks (pages 2–4)

- Attempt questions 1–10 using the multiple-choice answer sheet provided
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–10)

- Attempt questions 11–14, answer the questions in the writing booklets
- Allow about 1 hour and 45 minutes for this section

Section I (10 marks)	**Multiple Choice**	**/10**
Section II (60 marks)	**Question 11**	**/16**
	Question 12	**/14**
	Question 13	**/16**
	Question 14	**/14**
	Total	**/60**

NESA Number: .... Teacher: ....

Section I

10 marks
Attempt Questions 1 – 10
Allow 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 The coefficient of x^3 in the binomial expression $(x-2)^{10}$ is:

A. -15360

B. -11520

C. 11520

D. 15360

2 Which integral is obtained when the substitution $u = e^x$ is applied to $\int_0^1 \frac{e^x}{1+e^{2x}}dx$?

A. $\int_0^1 \frac{du}{1+u^2}$

B. $\int_1^e \frac{du}{1+u}$

C. $\int_0^1 \frac{du}{1+u}$

D. $\int_1^e \frac{du}{1+u^2}$

3 The multiplicity of the negative root of the polynomial below is:

$$y = (x-3)^2(x-2)(x+4)^4$$

A. 1

B. 2

C. 3

D. 4

4 The scalar projection of the vector $\underset{\sim}{a} = \underset{\sim}{i} + 4\underset{\sim}{j}$ in the direction of the vector $\underset{\sim}{b}$ is $\frac{11}{\sqrt{13}}$.

Which of the following could be equal to $\underset{\sim}{b}$?

A. $2\underset{\sim}{i} + 3\underset{\sim}{j}$

B. $3\underset{\sim}{i} + 2\underset{\sim}{j}$

C. $3\underset{\sim}{i} + 4\underset{\sim}{j}$

D. $4\underset{\sim}{i} + 3\underset{\sim}{j}$

5 What is the domain and range of the function $f(x) = 3\cos^{-1}\frac{x}{2}$?

A. D: $[-2,2]$, R: $[0,\frac{\pi}{3}]$

B. D: $[-2,2]$, R: $[-3,3]$

C. D: $[0,2]$, R: $[0,3\pi]$

D. D: $[-2,2]$, R: $[0,3\pi]$

6 If $\cos\alpha = -\frac{3}{4}$, where $\frac{\pi}{2} \leq \alpha \leq \pi$, what is the exact value of $\sin 2\alpha$?

A. $\dfrac{-3\sqrt{7}}{8}$

B. $\dfrac{3\sqrt{7}}{8}$

C. $\dfrac{15}{8}$

D. $\dfrac{30}{8}$

7 Given that $x = 40t$ and $y = 56t - 16t^2$, then the expression for $\frac{dy}{dx}$ is given by:

A. $\dfrac{dy}{dx} = \dfrac{5}{7-4t}$

B. $\dfrac{dy}{dx} = \dfrac{7-4t}{5}$

C. $\dfrac{dy}{dx} = 5(7-4t)$

D. $\dfrac{dy}{dx} = \dfrac{1}{5(7-4t)}$

8 Out of 10, 6 are to be selected for the final round of a competition. 4 of those 6 will be placed as 1st, 2nd, 3rd and 4th.

In how many ways can this process be carried out?

A. $\dfrac{10!}{6!4!}$

B. $\dfrac{10!}{4!4!}$

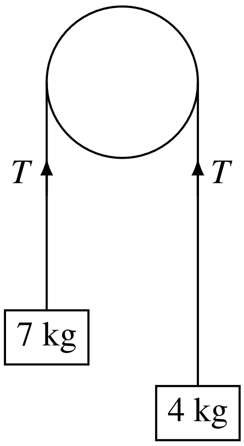
C. $\dfrac{10!}{4!2!}$

D. $\dfrac{10!}{6!}$

9 The diagram below shows a pulley with two objects of mass 7 kg and 4 kg attached to the ends of a light, inelastic string that passes through a smooth pulley.

The tension in the string is T newtons.

Take $g = 9.8$ m·s^{-2}



The downwards acceleration of the 7 kg, to 2 significant figures, is:

A. 2.7 m·s^{-2}

B. 9.8 m·s^{-2}

C. 11 m·s^{-2}

D. 35 m·s^{-2}

10 When added to water, 10 grams of salt dissolves at a rate equal to 15% of the amount of undissolved salt per hour.

If x is the number of grams of undissolved salt after t hours, then x satisfies the differential equation:

A. $\dfrac{dx}{dt} = -\dfrac{1}{15}x$

B. $\dfrac{dx}{dt} = -\dfrac{1}{10}x$

C. $\dfrac{dx}{dt} = -\dfrac{1}{10}(15 - x)$

D. $\dfrac{dx}{dt} = -\dfrac{1}{15}(10 - x)$

End of Multiple Choice

Section II

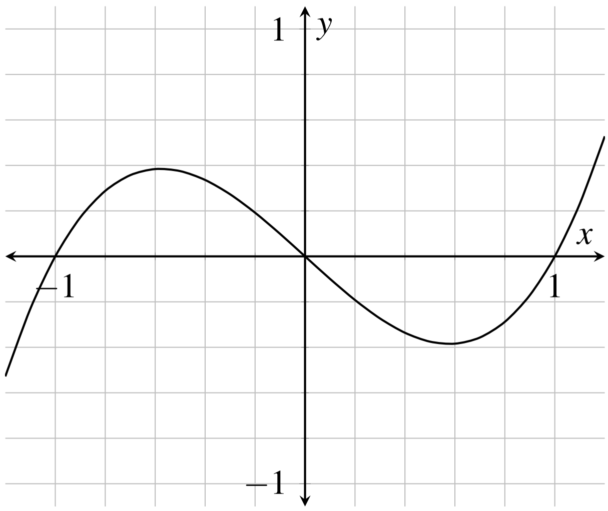
60 marks
Attempt Questions 11 – 14
Allow about 1 hour 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your response should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) The diagram below shows the graph of a function $f(x)$. The function has zeroes at $x = -1$ and $x = 1$, and a y-intercept at $y = 0$ as shown. **4**



In your writing booklet, sketch the graphs of $y = f(x)$, $y = \frac{1}{f(x)}$ and $y = |f(x)|$ on the same axes. Show all essential features, together with the location of the points corresponding to the labelled points on $y = f(x)$.
Your graph should be half a page.

(b) Find the equation of the curve $r = r(\theta)$, which satisfies the differential equation **4**

$$\frac{dr}{d\theta} = \frac{r^2}{\theta}$$

With the initial conditions $r(1) = 2$.

Question 11 continues next page

(c) Triangle ABC has vertices $A(-4,7)$, $B(3,6)$ and $C(-5,0)$.

(i) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in column vector form. **2**

(ii) Find the magnitude of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. **2**

(iii) Find $\angle BAC$. **2**

(iv) Hence calculate the area of the triangle. **1**

End of question 11

Question 12 (14 marks)

(a) Show that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$ **4**

(b) (i) Explain why $f(x) = x^2 - 3x$ does not have an inverse function. **1**

(ii) Find the largest possible domain, that includes $x = 0$, for the inverse of $f(x)$. **1**

(iii) Find an expression for $f^{-1}(x)$. **2**

(c) In how many way can 5 people be selected from a group of 12 if:

(i) one particular person must be included. **1**

(ii) two particular people cannot be selected. **1**

(iii) one particular person must be included, and two particular people are excluded from the selection. **1**

(d) When $P(x)$ is divided by $(x+2)$ the remainder is -4 and when $P(x)$ is divided by $(x+1)$ the remainder is 3. **3**
Find the remainder when $P(x)$ is divided by $(x+2)(x+1)$.

End of question 12

Question 13 (15 marks)

(a) Show that **3**

$$\int_{\frac{\pi}{8}}^{\frac{5\pi}{4}} \sin^2 3x\,dx = \frac{9\pi}{16} + \frac{1}{12} + \frac{\sqrt{2}}{24}$$

(b) Use mathematical induction to show that for all integers $n \geq 1$, **3**

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

(c) The equation $I(t) = 4\sin 0.02t + 8\cos 0.02t$ measures the current, I amperes, in a particular circuit after t seconds.

(i) Express the function in the form $I(t) = A\sin(at+b)$, for $0 \leq b \leq \frac{\pi}{2}$. **3**

(ii) When does the current first reach its maximum value? **3**

(d) One Spring Tuesday morning in Term 4, at 8 : 30 am, Vrund was on the Penrith High School hockey fields playing footy with the boys. He then kicked the football to Dev with all his might. If Vrund kicked the football from O, on level ground with velocity 25 m·s^{-1} at an inclination of θ, the equations of motion of the football are:

$$s(t) = 25t\cos\theta\,\underset{\sim}{i} + (25t\sin\theta - 4.9t^2)\,\underset{\sim}{j} \qquad \text{(DO NOT prove this)}$$

Unfortunately, Vrund used too much power and the football smashed the window of A.2.2 when it reached its maximum height.

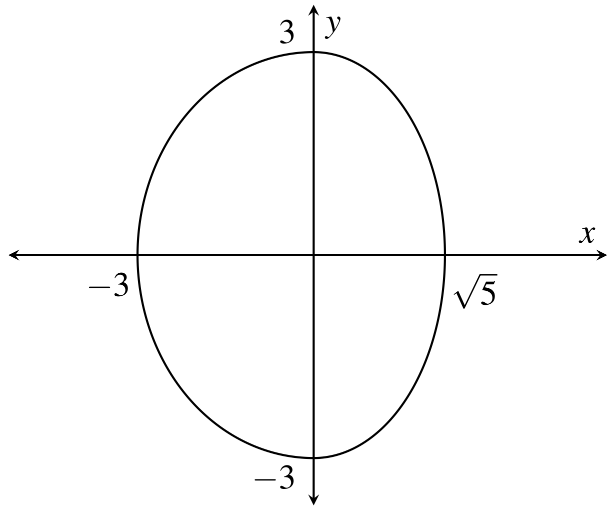
(i) Show that the football smashes the window at a height of $\frac{3125}{98}\sin^2\theta$ m. **2**

(ii) Show that the football hits the window at a horizontal distance of $\frac{3125}{98}\sin 2\theta$ metres from O. **1**

End of question 13

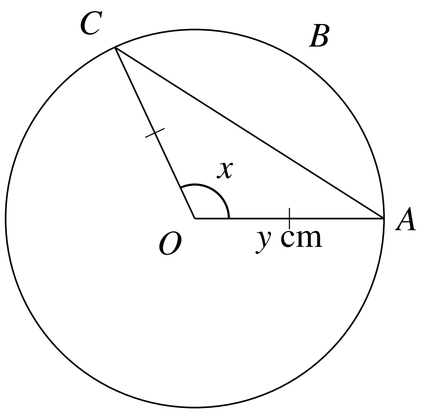
Question 14 (16 marks)

(a) The diagram shows two curves C_1 and C_2. The curve C_1 is an ellipse with the equation $\frac{x^2}{5}+\frac{y^2}{9}=1$, $x>0$ and C_2 is a semicircle $x^2+y^2=9$, $x \le 0$. **3**



A rattle is modelled by rotating the curves about the x-axis to form a solid of revolution. Show that the exact value of the volume of the solid of revolution is $\pi(6\sqrt{5}+18)\text{units}^3$.

(b) The chord AC cuts off a minor segment ABC in a circle with centre O and radius y cm. $\angle AOC$ is x radians.



It is given that y and x vary so that the area of the minor segment is constant at 50 cm^2.

(i) Show that $y=\dfrac{10}{\sqrt{x-\sin x}}$ **3**

(ii) If x is increasing at a rate of 0.05 radians per second, find the rate at which the radius is decreasing when $x=1.5$ radians, correct to 2 decimal places. **3**

Question 14 continues next page

(c) A fast-food restaurant has estimated that if they spend x on advertising their new product, it will attract a proportion $y(x)$ of the potential customers for the product, where

$$\frac{dy}{dx} = ay(1-y)$$

where $a > 0$ is a given constant.

(i) Explain why $\frac{dy}{dx}$ has its maximum value when $y = \frac{1}{2}$. **1**

(ii) Show that $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$ **1**

(iii) Using part (ii) or otherwise, deduce that **4**

$$y(x) = \frac{1}{ke^{-ax}+1}$$

for some constant $k > 0$.

(iv) The fast-food restaurant knows that if they spend no money on advertising the product then the number of customers will be one-fifth of the potential customers. Hence, find the value of the constant k referred to in part (iii). **1**

End of question 14

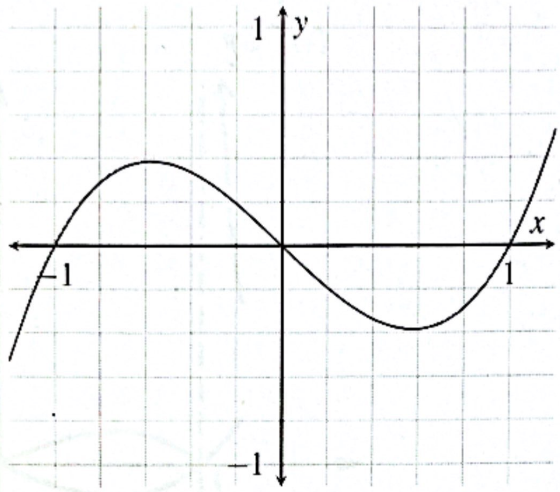
End of paper

Multiple-choice questions
Q.1) A Q.2) D Q.3) D Q.4) B Q.5) D
Q.6) A Q.7) B Q.8) C Q.9) A Q.10) B

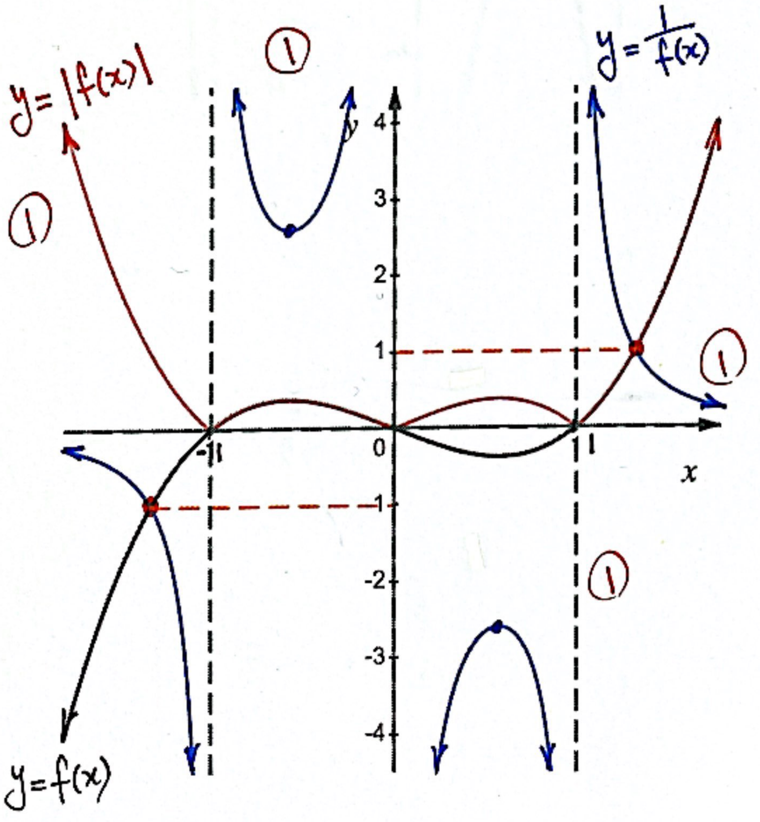
Poorly done

Question 11 (15 marks)

(a) The diagram below shows the graph of a function $f(x)$. The function has zeroes at $x=-1$ and $x=1$, and a y-intercept at $y=0$ as shown. **4**



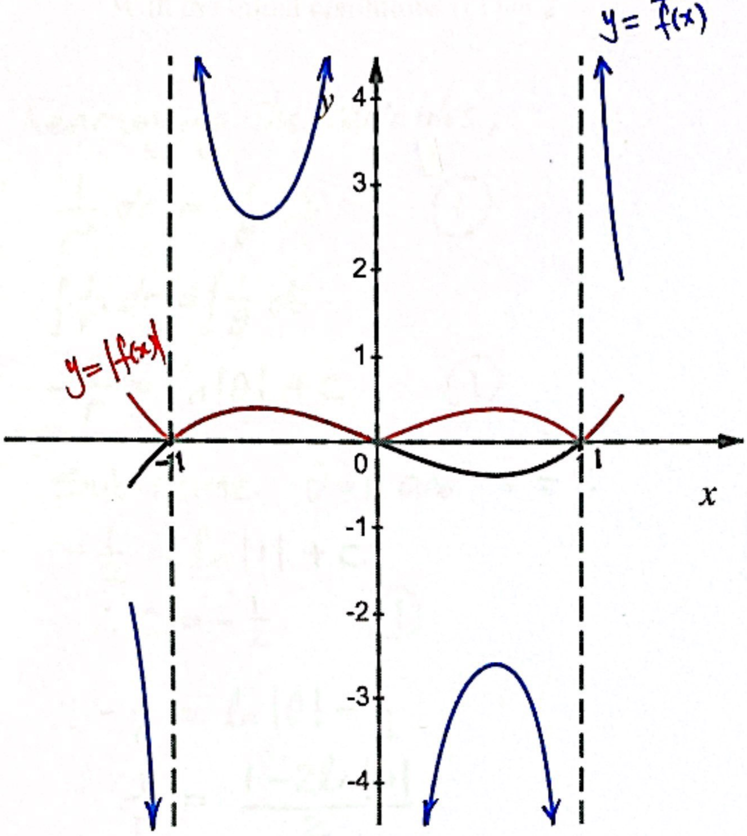
In your writing booklet, sketch the graphs of $y=f(x)$, $y=\frac{1}{f(x)}$ and $y=|f(x)|$ on the same axes. Show all essential features, together with the location of the points corresponding to the labelled points on $y=f(x)$.
Your graph should be half a page.



* Did not sketch $y=f(x)$
* $y=f(x)$ and $y=\frac{1}{f(x)}$ intersect at points where $y=1$ or -1.
* The two end parts of $y=|f(x)|$ must not be straight or concave down and no turning points at the x-intercepts.
* Turning points of $y=\frac{1}{f(x)}$ must be above $y=1$ and below $y=-1$.
* Some students sketched the three graphs separately and did not show where $y=f(x)$ and $y=\frac{1}{f(x)}$ intersect.
* Most students showed the asymptotes at $x=1$ and $x=-1$ but not at $x=0$ and $y=0$. Did not penalise if the graph of $y=\frac{1}{f(x)}$ was approaching all asymptotes.

Question 11 (a)

Also accepted:



Done well

Q.11)

(b) Find the equation of the curve $r = r(\theta)$, which satisfies the differential equation **4**

$$\frac{dr}{d\theta} = \frac{r^2}{\theta}$$

With the initial conditions $r(1) = 2$.

Rearranging the variables,

$$\frac{1}{r^2}\,dr = \frac{1}{\theta}\,d\theta \quad (1)$$

$$\int \frac{1}{r^2}\,dr = \int \frac{1}{\theta}\,d\theta$$

$$-\frac{1}{r} = \ln|\theta| + c \quad (1)$$

Substitute $\theta = 1$ and $r = 2$

$$-\frac{1}{2} = \ln|1| + c$$

$$\therefore c = -\frac{1}{2} \quad (1)$$

$$\therefore -\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

$$\frac{1}{r} = \frac{1 - 2\ln|\theta|}{2}$$

$$r = \frac{2}{1 - 2\ln|\theta|}$$

$$\therefore r = \frac{2}{1 - \ln\theta^2} \quad (1)$$

C/M

* Substituted $\theta = 2$ and $r = 1$ (Penalised)

C/M (Penalised)

* If $\frac{1}{r} = \frac{1}{2} - \ln|\theta|$ then $r \neq 2 - \frac{1}{\ln|\theta|}$

* Did not simplify fully.

$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$ or $r = \frac{2}{1 - 2\ln|\theta|}$

(No penalty)

Alternative

$$\frac{1}{r^2}\,dr = \frac{1}{\theta}\,dr$$

$$\int_2^r \frac{1}{r^2}\,dr = \int_1^\theta \frac{1}{\theta}\,dr$$

$$\left[-\frac{1}{r}\right]_2^r = \left[\ln|\theta|\right]_1^\theta$$

$$-\frac{1}{r} + \frac{1}{2} = \ln|\theta| - \ln 1$$

$$\frac{1}{r} = \frac{1}{2} - \ln|\theta|$$

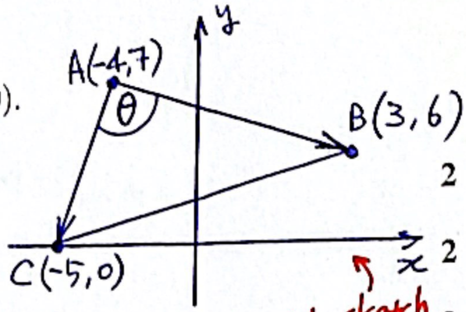
$$\frac{1}{r} = \frac{1 - 2\ln|\theta|}{2}$$

$$\therefore r = \frac{2}{1 - \ln\theta^2}$$

Done well

Q.11)

(c) Triangle ABC has vertices $A(-4,7)$, $B(3,6)$ and $C(-5,0)$.



(i) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in column vector form. 2

(ii) Find the magnitude of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. 2

(iii) Find $\angle BAC$. 2

(iv) Hence calculate the area of the triangle. 1

To avid, sketch

C/M
* $\overrightarrow{AB} = \overrightarrow{OB} + \overrightarrow{OA}$
$\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB}$
(Penalised)

i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= \begin{pmatrix} 3 \\ 6 \end{pmatrix} - \begin{pmatrix} -4 \\ 7 \end{pmatrix}$ $\quad = \begin{pmatrix} -5 \\ 0 \end{pmatrix} - \begin{pmatrix} -4 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ -1 \end{pmatrix}$ 1 $\quad = \begin{pmatrix} -1 \\ -7 \end{pmatrix}$ 1

ii) $|\overrightarrow{AB}| = \sqrt{7^2 + (-1)^2}$, $|\overrightarrow{AC}| = \sqrt{(-1)^2 + (-7)^2}$

$= \sqrt{50}$ $\quad = \sqrt{50}$

$= 5\sqrt{2}$ 1 $\quad = 5\sqrt{2}$ 1

Alternative
* Some students worked out $|\overrightarrow{BC}|$, then used cosine rule for $\angle BAC$. (Longer method)

iii) $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}| \cos\theta$

$-7 + 7 = 5\sqrt{2} \times 5\sqrt{2} \times \cos\theta$ 1

$0 = 50\cos\theta$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$ (θ is angle in the triangle, $\theta < \pi$)

$\therefore \angle BAC = \frac{\pi}{2}$ radians 1

C/M
* $\overrightarrow{AB} \cdot \overrightarrow{AC} = -7 + 7$
$= 0$
$\therefore \angle BAC = 90°$ without showing any steps.
(Penalised)

iv) Area $= \frac{1}{2}bh$ (Since $\angle BAC = 90°$)

$= \frac{1}{2} \times 5\sqrt{2} \times 5\sqrt{2}$

$= 25$ units^2 1

Alternative

Area $= \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 5\sqrt{2} \times 5\sqrt{2} \times \sin\frac{\pi}{2}$

$= 25$ units^2

Q 12 a)

$\tan\left(\frac{\pi}{4}\right) = 1$

$\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

where $\theta = \frac{\pi}{8}$

$\Rightarrow \tan\left(\frac{\pi}{4}\right) = \dfrac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$ 1

$1 = \dfrac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$

$\tan^2\left(\frac{\pi}{8}\right) + 2\tan\left(\frac{\pi}{8}\right) - 1 = 0$

let $u = \tan\left(\frac{\pi}{8}\right)$

1 $u^2 + 2u - 1 = 0$

$u = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2}$

$= \dfrac{-2 \pm 2\sqrt{2}}{2}$

1 $= -1 \pm \sqrt{2}$

1 Since $\frac{\pi}{8}$ is acute $\therefore \tan\left(\frac{\pi}{8}\right) > 0$

$\therefore \tan\left(\frac{\pi}{8}\right) = -1 + \sqrt{2}$

or

$t = \tan\left(\frac{\pi}{4\times 2}\right)$

$\therefore \tan\left(\frac{\pi}{4}\right) = \dfrac{2t}{1-t^2}$

$1 = \dfrac{2t}{1-t^2}$

$t^2 + 2t - 1 = 0$

Done very poorly

- Students tried to substitute exact values into RHS and got lost in algebra
- Students did $\tan\left(\frac{\pi}{8}\right) = \dfrac{2\tan\left(\frac{\pi}{16}\right)}{1-\tan^2\left(\frac{\pi}{16}\right)}$ and got lost.

Q12 bi) Since it's a many to one function, the horizontal line test will fail which means it doesn't have an inverse.

Done quite well

ii) $D_f : (\infty, \frac{3}{2}]$ $\Rightarrow$ $D_{f^{-1}} : [-\frac{9}{4}, \infty)$

$R_f : [-\frac{9}{4}, \infty)$ $\quad$ $R_{f^{-1}} : (\infty, \frac{3}{2}]$

Many silly mistakes
- Students found the domain of $f(x)$ not $f^{-1}(x)$.
- Some students stated the range, not the domain.

iii) Let $y = x^2 - 3x$

the inverse: $x = y^2 - 3y$ (1)

$$x + \frac{9}{4} = y^2 - 3y + \frac{9}{4}$$

$$x + \frac{9}{4} = \left(y - \frac{3}{2}\right)^2$$

$$\pm\sqrt{x + \frac{9}{4}} = y - \frac{3}{2}$$

$$y = \frac{3}{2} \pm \sqrt{x + \frac{9}{4}}$$

Done poorly.
- students need to review how to complete the square

$x = y(y-3)$

$\therefore y = \frac{x}{y-3}$ was common

- also students need to determine if

$y = \frac{3}{2} + \sqrt{x + \frac{9}{4}}$

or

$y = \frac{3}{2} - \sqrt{x + \frac{9}{4}}$

Since it includes $(0,0)$

$$0 = \frac{3}{2} \pm \sqrt{0 + \frac{9}{4}}$$

$$0 = \frac{3}{2} \pm \frac{3}{2}$$

it must be

$$y = \frac{3}{2} - \sqrt{x + \frac{9}{4}} \quad (1)$$

12 c i) $^{11}C_4 = 330$ Done well

ii) $^{10}C_5 = 252$ Done well

iii) $^{9}C_4 = 126$ Done well.

12 d)

$$P(x) = (x+2)(x+1)\,Q(x) + ax + b \qquad 1$$

$$P(-1) = -2a + b = -4$$

$$P(-1) = -a + b = 3$$

$$1 - 2: \quad \begin{array}{r} -2a + b = -4 \\ -(-a + b = 3) \\ \hline -a = -7 \\ a = 7 \end{array}$$

Sub $a = 7$ into 2

$$-7 + b = 3$$
$$b = 10$$

1

$\therefore$ Remainder is $7x + 10$ 1

Done very poorly

- Students recognised $P(-2) = -4$ and $P(-1) = 3$ but that's all so they guessed the remainder $= -12$.
- Some students assumed $P(x)$ was a cubic which just guaranteed time got wasted.

Question 13 (15 marks)

(a) Show that **3**

$$\int_{\frac{\pi}{8}}^{\frac{5\pi}{4}} \sin^2 3x\,dx = \frac{9\pi}{16} + \frac{1}{12} + \frac{\sqrt{2}}{24}$$

Identity needed:

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$ found on reference.

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$$\int_{\frac{\pi}{8}}^{\frac{5\pi}{4}} \sin^2 3x \, dx = \int_{\frac{\pi}{8}}^{\frac{5\pi}{4}} \frac{1}{2} - \frac{1}{2}\cos 6x \, dx$$

✓ 1 mark change identity

$$= \left[\frac{1}{2}x - \frac{1}{12}\sin 6x\right]_{\frac{\pi}{8}}^{\frac{5\pi}{4}}$$

✓ 1 mark integration.

$$= \left(\frac{5\pi}{8} - \frac{1}{12}\sin\left(\frac{15\pi}{2}\right)\right) - \left(\frac{\pi}{16} - \frac{1}{12}\sin\left(\frac{3\pi}{4}\right)\right)$$

$$= \left(\frac{5\pi}{8} + \frac{1}{12}\right) - \left(\frac{\pi}{16} - \frac{\sqrt{2}}{24}\right)$$

✓ 1 mark substitution and solve.

$$= \frac{9\pi}{16} + \frac{1}{12} + \frac{\sqrt{2}}{24}$$

Question was poorly done.

- Many students not able to recognise integral of $\sin^2\theta$.
- Common error with integral of $-\frac{1}{2}\cos 6x$.

- Common error with grouping symbols and negatives.

(b) Use mathematical induction to show that for all integers $n \geq 1$, **3**

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Step 1: Let $n=1$

LHS $= 1$ RHS $= \frac{1(1+1)}{2}$

$= 1$

$\therefore$ LHS = RHS, true for $n=1$.

Step 2: Assume true for $n=k$, $k \geq 1$

$$1+2+\ldots+k = \frac{k(k+1)}{2}$$

✓ 1 mark for step 1 and step 2

Step 3: Prove true for $n=k+1$

$$1+2+\ldots+k+(k+1) = \frac{(k+1)(k+1+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

LHS $= 1+2+\ldots+k+(k+1)$ ✓ 1 mark for using assumption

$= \frac{k(k+1)}{2} + (k+1)$ (from step 2 assumption)

$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{k(k+1)+2(k+1)}{2}$ ✓ 1 mark algebra

$= \frac{(k+1)(k+2)}{2}$

$=$ RHS

Therefore, using the principle of mathematical induction, the result is true for all $n \geq 1$.

(c) The equation $I(t) = 4\sin 0.02t + 8\cos 0.02t$ measures the current, I amperes, in a particular circuit after t seconds.

(i) Express the function in the form $I(t) = A\sin(at + b)$, for $0 \le b \le \frac{\pi}{2}$. **3**

$I(t) = 4\sin 0.02t + 8\cos 0.02t$

$I(t) = A\sin at\cos b + A\cos at\sin b$

Equate the expressions

$a = 0.02$ $\quad$ $A\cos b = 4$ $\quad$ ✓ 1 mark

$A\sin b = 8$ $\quad$ set up and equating

$A^2\cos^2 b + A^2\sin^2 b = 4^2 + 8^2$

$A^2(\cos^2 b + \sin^2 b) = 4^2 + 8^2$

$A^2(1) = 4^2 + 8^2$

$A = \pm\sqrt{80}$

since $A > 0$ $\quad$ ✓ 1 mark

$\therefore A = 4\sqrt{5}$ $\quad$ for A

$\dfrac{\sin b}{\cos b} = \dfrac{8}{4}$

$\tan b = 2$ $\quad$ ✓ 1 mark

$b = \tan^{-1}(2)$ $\quad$ for b

$\therefore I(t) = 4\sqrt{5}\sin(0.02t + \tan^{-1}(2))$

Better to leave in exact value

- Poorly done, many students still not able to recognise auxiliary angle questions.

(ii) When does the current first reach its maximum value? **3**

A sine graph first reaches its max value at $\frac{\pi}{2}$. ✓ 1 mark

For $I(t) = 4\sqrt{5}\sin(0.02t + \tan^{-1}(2))$

$$0.02t + \tan^{-1}(2) = \frac{\pi}{2}$$

$$0.02t = \frac{\pi}{2} - \tan^{-1}(2)$$ ✓ 1mark

$$t = \frac{\frac{\pi}{2} - \tan^{-1}(2)}{0.02}$$

✓ 1 mark

$t = 23$ seconds, first reaches max at 23 seconds.

(d) One Spring Tuesday morning in Term 4, at 8 : 30 am, Vrund was on the Penrith High School hockey fields playing footy with the boys. He then kicked the football to Dev with all his might. If Vrund kicked the football from O, on level ground with velocity 25 m·s^{-1} at an inclination of θ, the equations of motion of the football are:

$$s(t) = 25t\cos\theta\,\underset{\sim}{i} + (25t\sin\theta - 4.9t^2)\,\underset{\sim}{j} \qquad \text{(DO NOT prove this)}$$

Unfortunately, Vrund used too much power and the football smashed the window of A.2.2 when it reached its maximum height.

(i) Show that the football smashes the window at a height of $\frac{3125}{98}\sin^2\theta$ m. **2**

$s(t) = 25t\cos\theta\,\underset{\sim}{i} + (25t\sin\theta - 4.9t^2)\,\underset{\sim}{j}$

Football breaks window at max height.
Calculate max height when vertical velocity = 0.

$y = 25t\sin\theta - 4.9t^2$ (vertical displacement)

$\dot{y} = 25\sin\theta - 9.8t$

$0 = 25\sin\theta - 9.8t$

$9.8t = 25\sin\theta$

$t = \dfrac{25\sin\theta}{9.8}$

✓ 1 mark for differentiation and calculate t.

Sub t into vertical displacement.

$y = 25\left(\dfrac{25\sin\theta}{9.8}\right)\sin\theta - 4.9\left(\dfrac{25\sin\theta}{9.8}\right)^2$

✓ sub t and solve for y.

$= \dfrac{625\sin^2\theta}{9.8} - \dfrac{3125\sin^2\theta}{98}$

$y = \dfrac{3125\sin^2\theta}{98}\ \sin^2\theta$ m

(ii) Show that the football hits the window at a horizontal distance of $\frac{3125}{98}\sin 2\theta$ metres from O. **1**

Sub t into horizontal displacement.

$$x = 25\left(\frac{25\sin\theta}{9.8}\right)\cos\theta$$

$$= \frac{25^2\sin\theta\cos\theta}{9.8}$$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$$= \frac{625}{9.8} \times \frac{1}{2}\sin 2\theta$$

✓ 1 mark for sub and solve

$$= \frac{3125}{98}\sin 2\theta$$

identify $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ is necessary.

2022 Mathematics Ext. 1 TRIAL PAPER

Question 14

(a) Ellipse

$$\left(\frac{x^2}{5}+\frac{y^2}{9}=1\right)\times 45$$

$$9x^2+5y^2=45$$

$$5y^2=45-9x^2$$

$$y^2=9-\frac{9}{5}x^2$$

SemiCircle

$$x^2+y^2=9 \quad , x\le 0$$

$$y^2=9-x^2$$

$$\text{Volume} = \pi\int_{-3}^{0}(9-x^2)\,dx + \pi\int_{0}^{\sqrt{5}}\left(9-\frac{9}{5}x^2\right)dx \quad ✓$$

$$= \pi\left[9x-\frac{x^3}{3}\right]_{-3}^{0} + \pi\left[9x-\frac{3x^3}{5}\right]_{0}^{\sqrt{5}} \quad ✓$$

$$= \pi\left[0-\left(9(-3)-\frac{(-3)^3}{3}\right)\right] + \pi\left[\left(9\sqrt{5}-\frac{3}{5}(\sqrt{5})^3\right)-0\right] \quad ✓$$

$$= 18\pi + \pi(6\sqrt{5})$$

$$= \pi(6\sqrt{5}+18) \text{ units}^3$$

1 mark for correct definite integrals with correct limits

1 mark for correct integration

1 mark for showing correct substitution of values

Common errors:

- Students mixed up the limits for ellipse and semicircle
- Students did not show the substitution of values
- Students used incorrect formula for volume.

Brilliant idea:

- Students recognised that the volume of a hemisphere is $\frac{1}{2}\times\frac{4}{3}\pi r^3$.

Question 14

(b)

(i) Area of segment = Area of sector − Area of triangle

$$50 = \frac{1}{2}y^2x - \frac{1}{2}(y)(y)\sin x$$ ✓

$$50 = \frac{1}{2}y^2(x - \sin x)$$

$$100 = y^2(x-\sin x)$$

$$y^2 = \frac{100}{x-\sin x}$$ ✓

$$y = \pm\sqrt{\frac{100}{x-\sin x}}$$ but $y>0$ since it is the length of a radius ✓

$$\therefore y = \frac{10}{\sqrt{x-\sin x}}$$

To achieve 3 marks, students need to show that $y = \pm\sqrt{\frac{100}{x-\sin x}}$ and explain why $y>0$.

Common errors: Student using area of sector = 50 or area of triangle = 50.
Students did not explain why $y = \pm\sqrt{\frac{100}{x-\sin x}}$ and $y>0$.

(ii) $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

where $\frac{dx}{dt} = 0.05$ rad/s

$y = 10(x-\sin x)^{-\frac{1}{2}}$

$\frac{dy}{dx} = -5(x-\sin x)^{-\frac{3}{2}}(1-\cos x)$ ✓

when $x = 1.5$ rad

$$\frac{dy}{dt} = -5(1.5-\sin 1.5)^{-\frac{3}{2}}(1-\cos 1.5)\times 0.05$$ ✓

$= -0.65218\ldots$

$= -0.65$ cm/s ✓

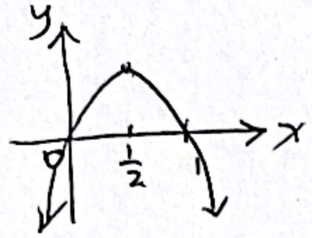
$\therefore$ radius is decreasing at a rate of 0.65 cm/s

Common errors: − Students solved $\frac{dy}{dx}$ incorrectly. Some using quotient rule or a failure to use the chain rule correctly.
− Students considering 1.5 rad as 1.5π rad.

Question 14

(c)

(i) The graph of $\frac{dy}{dx} = ay(1-y)$ is a concave down parabola.



Maximum value happens when

$y = \frac{0+1}{2} = \frac{1}{2}$

1 mark for explaining correctly.

(ii) $LHS = \frac{1}{y} + \frac{1}{1-y}$

$= \frac{1-y+y}{y(1-y)}$

$= \frac{1}{y(1-y)} = RHS$

1 mark for showing LHS = RHS

(iii)
$$\frac{dy}{dx} = ay(1-y)$$
$$\frac{dy}{y(1-y)} = a\,dx$$
$$\int\left(\frac{1}{y}+\frac{1}{1-y}\right)dy = \int a\,dx \quad \text{(from (ii))}$$
1 mark

$$\ln|y| - \ln|1-y| = ax + c$$
$$\ln|1-y| - \ln|y| = -ax - c$$
$$\ln\left|\frac{1-y}{y}\right| = -ax - c$$
1 mark

$$\frac{1-y}{y} = e^{-ax-c}$$
$$\frac{1-y}{y} = e^{-ax}e^{-c}$$
1 mark

let $e^{-c} = K$ where $K > 0$

$$\frac{1-y}{y} = Ke^{-ax}$$
$$\frac{1}{y} - 1 = Ke^{-ax}$$
$$\frac{1}{y} = Ke^{-ax} + 1$$
$$\therefore y = \frac{1}{Ke^{-ax}+1}$$
1 mark

Common error: Students simplifying incorrectly and not defining what K represents.

(iv) $y(x) = \frac{1}{Ke^{-ax}+1}$

When $x = 0$, $y(x) = \frac{1}{5}$

$\frac{1}{5} = \frac{1}{K+1}$

$\therefore K = 4$ ✓ 1mark